**Various Mechanical Examples**

I’d like to work out a few examples to illustrate how our results can be used, and to place the problem solving methodology in the context of problems solved in the past with N2L, or WE equation. The solution method is hopefully very general. But won’t be the most concise. But I’m often confused by the jumble of equations at my disposal so hopefully this method will take the guess work out, though at the expense perhaps of brevity.

What we are typically attempting to do is start with some known state of the system (usually specifying E0, X0 ={V0, **P**0, **L**0}, N0 or their conjugates), apply some process to it whereby we do work, add heat, or particles, apply a force, torque, etc., and subsequently attempt to determine the final state of the system E, X = {V, **P**, **L**}, N or their conjugates. The information provided is usually a mix of initial state information, process information, and final state information. At the fundamental level, every problem ought to be solvable by writing and solving the following equations:

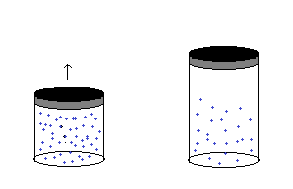
1. State known initial variables E, X, N, ψ. If know their conjugates instead, then parlay that into information about E, X, N, ψ using knowledge of S(E,X,N,ψ). For instance for the ideal gas we have S(U,V) = Nkln[(V/NΦ)(2U/Nkf)f/2], and dS = (1/T)dU + (p/T)dV, which tells us that U = (f/2)NkT and p = (2/f)(U/V) = NkT/V. This presumes the gas is initially in equilibrium, though. Also note that the U and p equations are basically the heat capacity equation and the equation of state and we can treat knowing these as equivalent to knowing the full S equation I think.
2. Basically same as above. State known final variables E, X, N, ψ. If know their conjugates instead, then parlay that into information about E, X, N, ψ using knowledge of S(E,X,N,ψ). For instance for the ideal gas we have S(U,V) = Nkln[(V/NΦ)(2U/Nkf)f/2], and dS = (1/T)dE + (p/T)dV, which tells us that U = (f/2)NkT and p = (2/f)(U/V) = NkT/V. This presumes the gas is finally in equilibrium, though. Again, knowledge of U and p are equivalent to knowledge of S(U,V) itself I think.
3. Write out balance equations for each variable {E, X, N, ψ} and solve to the extent possible, in terms of whatever free variables, η, may be left over. Then we plug this into our entropy balance equation. Typically we will have a closed system, and so this will simply instruct us to maximize S w/r to the free parameters η. This should provide whatever information is lacking in our connection between initial and final states. Balance equations are, succinctly,



where **F**, **T**, W, Q are shorthand for all force, torque, work and heat terms. It also may be of interest to us to determine the time-dependence of the path from initial to final state. And this would require knowing the time-dependence of the inputs of the E, X, N, ψ balance equations, and we would also have to assume that the system is in quasi-equilibrium the entire time and so the process via which it changes is reversible.

**1. Example**

Suppose we have a gas at T0, V0 and we suddenly raise the lid adiabatically, so quickly that the gas doesn’t exert any pressure on the lid, to volume VL. What will be the new temperature, volume, of the gas after it equilibrates?



*Initial Conditions*

So our initial information is T0 and V0. Since the gas is in equilibrium initially we can also say that U0 = (f/2)NkT0. So now we have initial conditions for the only two d.o.f.

*Final Conditions*

We have no final conditions specified, except that the lid is raised to volume VL, but this doesn’t yet mean that the gas ends up at that volume.

*Balances*

Now we do the balances. The energy balance would yield:



Where π is the ‘actual’ pressure of the gas, as opposed to the equilibrium value, and Vℓ is the (instantaneous) volume of space enclosed by the lid. And so we have energy conservation. So our final U1 = U0. And we have ‘volume’ balance:



(where is just some unknown function that tells us how the volume of the gas changes) But we do not know what is, technically I’d say, nor even yet that it will be such as to lead the gas to maximum volume (for instance if gas were a solid, it wouldn’t rise to fill the container to VL necessarily). This information must come from the entropy equation I’d say. So all we can say is that V1 = V0 + ΔV, where ΔV = ∫dt. Speaking of, the entropy balance would yield:



The last line indicates that the internal entropy term will drive the system to maximum entropy, consistent with external constraints. So S = max. So filling the balances into our entropy equation we have:



And our entropy balance says that ΔV will be such as to maximize S, which means it will be such as to make the gas fill the volume (as can see higher V means higher S). So now we know the final V1 = VL. Coupled with our earlier conclusion that U1 = U0, we’ve determined the final state of the gas.

*Postlogue*

And we can parlay our knowledge of U and V into information about T using S(U,V). So we have our final T1 = 2U1/fNk = 2U0/fNk = T0. We can get the change in entropy too, now that we know initial and final U’s, V’s.

**2. Example**

Consider a gas starting at T0­, V0. We raise the container to volume VL, allowing the gas to do work W, absorb heat Q, which are presumed known. What would be the new temperature, volume of the gas after it equilibrates?

*Initial Conditions*

Again we can use the initial conditions to work out our initial conditions: U0 = fNkT0/2, and initial V = V0.

*Final Conditions*

We cannot say, yet, what the final state looks like.

*Balances*

The energy balance would yield:



Presuming we know W and Q, as we said, it follows from just the first line that our final U1 = U0 – W + Q. Volume balance would be:



And again all we can say here is that our final V1 = V0 + ΔV, but cannot yet say that ΔV = anything in particular. The entropy balance would yield:



This line indicates that the internal entropy term will drive the system to maximum entropy, consistent with external constraints, regardless of Q. So we have, filling our balances into S,



With U fixed to U1, we see that to maximize entropy we want ΔV as large as possible. So it will be such as to make V1 = V0 + ΔV = VL. So we have our final state: U1 = U0 – W + Q, and our final V1 = VL.

*Postlogue*

And we can parlay that information into stuff about temperature T1 = 2U1/fNk = (U1/U0). And if we wanted, we could get pressure, say from ∂S/∂V = p/T. And find: p1 = 2U1/fV1.

**3. Example**

Consider a gas starting at T0­, V0. We heat the gas isometrically and in quasi-equilibrium manner so that it’s temperature increases to T1. How much heat was poured in? What is the new pressure? What changes if the process isn’t quasi-equilibrium?

*Initial Conditions*

So we know initial T0 and V0 and so we know the initial state U0 = (f/2)NkT0, and V0.

*Final Conditions*

And we know the final state U1 = (f/2)NkT1, and V1 = V0, since we said we heat the gas isometrically. So we know completely the initial and final states. Follows that pressure is p1 = NkT1/V1.

*Balances*

But proceeding to the balances.



Integrating we have ΔU = Q, whatever that is. Volume balance would be



And so V1 = V0 of course. And for entropy,



Integrating both sides of this equation, we’d have: ΔS = ∫dQ/T, whatever that is. Now we can get Q from the energy balance: Q = U1 – U0 = (f/2)Nk(T1 – T0).

*Postlogue*

Finally, even if the process were non-equilibrium, entropy maximization would’ve restored the volume to V1 = V0, and still, no work would’ve been done. So nothing critical would’ve changed. This is because with volume fixed there can be no irreversible energy conversion (because there is no energy conversion *at all*), i.e., conversion of ordered energy to disordered energy.

**4. Example**

Consider a gas starting at T0­, V0. We raise the container slowly and adiabatically so that gas remains in equilibrium the entire time, to new volume VL. What will be the new temperature, pressure of the gas? How much work will have been done? What changes if gas isn’t in equilibrium the entire time?

*Initial Conditions*

As we argued before, we know the initial U0 = fNkT0/2 and V0.

*Final Conditions*

And we could preemptively argue that the volume of the gas finally will be V1 = VL, since it remains in equilibrium the entire time, and we know that the form of S(U,V) will be such as to make the gas expand to fill the volume. So we know half the final state. But without knowing something else, we cannot ascertain the final state entirely.

*Balances*

The energy balance would yield:



where in the last line we preemptively plug in a consequence of the entropy balance, namely that the gas is always in equilibrium and so π = p and Vℓ = V, the volume of the gas. Volume balance is as usual:



The entropy balance would yield:



Since S­int is always 0, the gas is always in equilibrium, and recalling the expression for S that again tells us that the volume enclosed by the lid is in fact the same as that of the gas (V = Vℓ), and that the gas’s pressure is in fact π = p, its equilibrium pressure.

So basically we just need to determine the new energy. There are two ways to do this. One is to use the energy balance equation, in conjunction with the fact that p = (2/f)(U/V). And so we can write:



And so then we have U1. Another way that bypasses the energy balance per se´ and just uses the entropy expression is this. Since entropy is constant, we have:



So again we get the new energy. So now we know the final state.

*Postlogue*

And we can parlay this into information about temperature and pressure: T1 = 2U1/fNk, using the entropy equation. And we can further conclude p1 = 2U1/fV1. From which both the pressure and temperature follow:



The work done would be simply W = ΔU, from the first law. But we could also integrate directly W = ∫pdV. But if gas hadn’t been in equilibrium the entire time, we wouldn’t have been able to say π = p, or ΔS = 0, and we wouldn’t have been able to determine new T, p, or W.

**5. Example**

Consider a gas starting at T0­, V0. We isothermally raise the container slowly so that gas remains in equilibrium the entire time till it gets to new lid volume VL. How much heat will have been absorbed? How much work will it have done? What would change if it were out of equilibrium?

*Initial Conditions*

So we know the initial U0 = fNkT0/2, and volume V0.

*Final Conditions*

And our final temperature is T1 = T0. And so we know our final energy U1 = (f/2)NkT1. And we can preemptively say that we know the new volume, since the gas remains in equilibrium the entire time, and we know that S(U,V) will make the gas expand to fill the available volume. So we can say: V1 = VL. So we know the complete initial and final state of our system.

*Balances*

Then our balances would yield,



Volume balance would be



And entropy balance,



To get W and Q we have two options. One is to evaluate work directly W = ∫pdV, and then get Q from the energy balance:



The other option is to use the entropy balance equation. We have that:



and therefore from the first law we have:



as well. What changes if process is non-equilibrium? Then ΔU and ΔS would still be the same because the gas would still be in final state U1 and V1 = VL, but two things would’ve prevented us from getting either W or Q. First we couldn’t say π = p and so couldn’t directly evaluate W, or get Q from 1st law thereby, though we could’ve still concluded W = Q, whatever it was. Second, the entropy balance would’ve had an extra Sint. term, and so we couldn’t have set ΔS equal to the Q integral, or obtained W as we did.

**6. Example**

Suppose have a cylinder of gas at T0, V0, and then I add heat Q at such a rate so as to keep its pressure constant (most reliably done with a volume reservoir) and in equilibrium. What would be the new temperature and volume of the gas? How much work would gas do? What changes if process is not quasi-equilibrium?

We start with knowledge of U0 = fNkT0/2 and V0. So we know the intial state. Incidentally, we then know our initial pressure p0 = 2U0/fV0. And we know our final pressure is also p1 = p0. But we need one more piece of information to get complete knowledge of our final state. Our balances would yield,



where I preempt the fact that the entropy balance will say everything is in equilibrium the entire time. Volume balance would be:



And entropy balance,



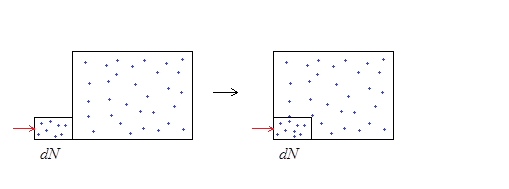
Filling stuff into the 1st law we have:



So that gives us the new U1 = U0 + Q/(1+2/f). And then the new volume follows: V1 = (2/f)U1/p1 = (2/f)U1/p0. And then the new temperature would follow from T1 = (2/f)U1/Nk. Then we could also calculate the work done. This follows from 1st law, but could also just use fact that gas is in equilibrium whole time, and so work would be W = ∫pdV = p­­0ΔV. If quasi-equilibrium fails, then we cannot equate π to p, which would prevent us from getting ΔU.

**7. Example**

Suppose that we have a container initially filled with gas at temperature Tin = T0, pressure pin = p0, and volume Vin = V0 (which wouldn’t change). And we are pumping gas with temperature Tout, pressure pout, and volume Vout into the container with a pressure equal to the interior pressure at rate of r = dV/dt (in m3/s), and at a speed vout. How long would it take for the container to fill up with gas to the maximum pressure our pump is able to exert, pmax? What would be the amount (N) and temperature (T) of the gas by the time the container was filled up? We’ll assume that after each piece of gas(out) is pushed through, the ensemble comes to equilibrium before another is pushed in, and that this equilibrium process happens relatively instantaneously.



A mass balance would yield:



where I is the particle current. This would be given by I = dN/dt = d(pV/kT)/dt = (pout/kTout)r. And energy balance:



And now the entropy balance:



There is definitely an internal entropy change as the small volume of gas dV expands to fill the entire container. So let’s determine the upshot of this irreversible process, which lasts a short time *dt* by our initial assumption. As the internal entropy increases, we will have:



since equilibrium will prevail before after, we can write the energy of the gas before and after in terms of its temperature. So we’ll have:



where N = N0 + Nout. Before we proceed, it’s probably best to write NkT as pinV0. And then we’ll have:



The solution of this equation is:



and we have:



and so we can solve this equation for when pin will equal pmax. And we can also figure out the temperature at that time.